Extended pole placement technique and its applications for targeting unstable periodic orbit

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In this paper we extend the pole placement technique in the case of one adjustable system parameter. The extended technique allows a more general choice of feedback forms, so it can improve the performance of the control system in many aspects. As an example of the application, we show how to target the unstable orbit from the corresponding stable one in the vicinity of a tangential bifurcation point without the need of knowing the location of the unstable orbit in advance. This technique can be used to obtain the unstable output related to bistability from an experimental device. [S1063-651X(98)10305-7]

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I. INTRODUCTION

Recently, much attention has been focused on stabilizing unstable periodic orbits of a nonlinear dynamical system in term of chaos control [1–7]. For the sake of simplicity, we consider a discrete time dynamical system that has one adjustable system parameter,

$$\mathbf{z}_{i+1}=\mathbf{f}(\mathbf{z}_i,p), \tag{1.1}$$

where $\mathbf{z}_i\in\mathbf{R}^n$, $p\in\mathbb{R}$, and $\mathbf{f}$ is sufficiently smooth in both variables. Here p is considered as the parameter that is available for external adjustment but is restricted to lie in some small interval $\|p-p_0\|<\delta$ around a nominal value p_0. Let $\mathbf{z}_*(p_0)$ be a fixed point of Eq. (1.1) (i.e., a period-one orbit of the map $\mathbf{f}$; the consideration of the periodic orbits of period larger than one is straightforward). For values of p close to p_0 and in the neighborhood of the orbit $\mathbf{z}_*(p_0)$, the map (1.1) can be approximated by the linear map

$$\delta\mathbf{z}_{i+1}=\mathbf{A}\delta\mathbf{z}_i+\mathbf{B}\delta p_i, \tag{1.2}$$

where $\delta\mathbf{z}_i=\mathbf{z}_i-\mathbf{z}_*(p_0)$, $\delta p_i=p_i-p_0$, and $\mathbf{A}=\mathbf{D}_z\mathbf{f}(\mathbf{z}_*,p_0)$ is an $n\times n$ Jacobian matrix and $\mathbf{B}=\mathbf{D}_p\mathbf{f}(\mathbf{z}_*,p_0)$ is an n-dimensional column vector. The most popular way to stabilize the fixed point $\mathbf{z}_*(p_0)$ is to use a linear feedback

$$\delta p_{i+1}=\mathbf{k}^T\cdot(\mathbf{z}_{i+1}-\mathbf{z}_*) \tag{1.3}$$

to the system, in which case Eq. (1.2) appears as

$$\delta\mathbf{z}_{i+1}=(\mathbf{A}+\mathbf{B}\mathbf{k}^T)\delta\mathbf{z}_i, \tag{1.4}$$

where $\mathbf{k}$ is a constant vector to be determined.

There are two methods to calculate the suitable $\mathbf{k}$. The first way, as has been pointed out by Romeiras *et al.* [3], is the "pole placement technique," which is well known from the control system theory (see, for example, [8]). This technique can be applied to choose suitable $\mathbf{k}$ with $\mathbf{A}$ and $\mathbf{B}$ given such that the eigenvalues (the "regulator poles") of the matrix $\mathbf{A}+\mathbf{B}\mathbf{k}^T$ have specified values. Those values of $\mathbf{k}$ that make the modulus of all the eigenvalues of $\mathbf{A}+\mathbf{B}\mathbf{k}^T$ smaller than unity form a region in R^n space (we call it the stability region in the following) and any point in this region can be used to stabilize the desired orbit.

The second way is the Ott-Grebogi-Yorke (OGY) method [1], by which one can obtain a special value of $\mathbf{k}$ to force the system trajectories to fall on the local stable manifold of $\mathbf{z}_*$ such that it makes $\mathbf{z}_*$ stable. The value of $\mathbf{k}$ is in fact a special point belonging to the stability region calculated from the pole placement technique [3]. This method has triggered immense research activities to apply feedback control to chaotic systems.

To apply the pole placement technique or the OGY method one only requires the location of the desired periodic orbit, the linearized dynamics about the periodic orbit, and the dependence of $\mathbf{f}$ on a small variation of the control parameter. Therefore, in principle, the above methods, using the delay coordinate embedding technique [9,10], can be applied to nonlinear experimental systems without any *a priori* knowledge of the systems of equations governing the dynamics [4].

Nevertheless, the linear feedback function (1.3) is constructed by the prompt information $\mathbf{z}_{i+1}$ and the desired orbit $\mathbf{z}_*$, which limit the application of the feedback control method. For example, in the case of experimental systems, especially in the absence of an *a priori* mathematical system model where the delay coordinates are used, it is difficult to determine $\mathbf{z}_*$ exactly. The situation will be too bad when one intends to track $\mathbf{z}_*$ in the case that some of the system parameters change as a function of time, since $\mathbf{z}_*$ cannot be determined in advance. In other situations, e.g., in the case that the time spent on the feedback circuit cannot be neglected, prompt feedback is not accessible.

Some of the present authors have shown [5] that there are in fact many other suitable feedback forms in addition to Eq. (1.3). Without losing generality, let

$$p_{i+1}=g(\mathbf{z}_i,p_i) \tag{1.5}$$

be the feedback function. The perturbed system is an $(n+1)$-dimensional system and the desired orbit appears as the

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fixed point $(\mathbf{z}_*,p_0)$ of this system. The linear approximation of the perturbed system in the neighborhood of the desired orbit is

$$\begin{pmatrix} \delta\mathbf{z}_{i+1} \\ \delta p_{i+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & D \end{pmatrix} \begin{pmatrix} \delta\mathbf{z}_i \\ \delta p_i \end{pmatrix}, \tag{1.6}$$

where $\mathbf{C}=\mathbf{D}_z g$ is an n-dimensional row vector and $D=\partial g/\partial p$ is a constant. Let

$$\mathbf{T}=\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & D \end{pmatrix}$$

denote the $(n+1)\times(n+1)$ Jacobian matrix. To stabilize $(\mathbf{z}_*,p_0)$ by applying Eq. (1.5) one needs to find the solution of $\mathbf{C}$ and D with $\mathbf{A}$ and $\mathbf{B}$ given such that the eigenvalues of $\mathbf{T}$ have specified values. When obtaining those solutions that make all the eigenvalues of $\mathbf{T}$ have modulus smaller than unity, the feedback function that satisfies $\mathbf{C}=\mathbf{D}_z g$ and $D=\partial g\partial p$ can all be applied to stabilize the desired orbit. However, this purpose cannot be approached by use of the pole placement technique or the OGY method as well as its direct extensions. Zhao *et al.* [5] have discussed how to choose various suitable feedback forms for different purposes, but a general solution to the problem of determining $\mathbf{C}$ and D has not been given.

The first purpose of this paper is to derive the equations of $\mathbf{C}$ and D such that the eigenvalues of $\mathbf{T}$ have specified values. Finishing this step, we actually approach a more general pole placement technique that allows us to use more general feedback forms for the control of the dynamical systems. Equations given in the present paper are suitable for either the case of knowing the dynamical system model or the situation of using the delay coordinate embedding technique without any *a priori* knowledge of the system of equations governing the dynamics; thus the application of the extended technique to experimental systems is direct.

As an example of applying the extended technique, another aim of the paper is to develop a method to target an unstable periodic orbit that is created at a tangential bifurcation point. At the time before targeting, the dynamical state trajectories lie on the stable periodic orbit created at the same tangential bifurcation point and the position of the unstable orbit is unknown. This method, combined with the tracking technique presented in Ref. [5], can be applied to realize the unstable output related to bistability from an experimental device.

The plan of this paper is as follows. In Sec. II we extend the pole placement technique. Coefficients in some useful feedback functions are also derived in this section. Without losing generality, the results in this section are obtained using delay coordinates. In Sec. III we first describe how to target the unstable orbit from the corresponding stable orbit created at the same tangential bifurcation point in the case of one-dimensional systems. A numerical example for obtaining and tracking the unstable state existing in a one-dimensional bistability system is given. We then extend the results to the case of high-dimensional systems and apply the results to the Hénon map. We summarize our results in Sec. IV.

II. EXTENSION OF THE POLE PLACEMENT TECHNIQUE

In experimental studies of chaotic dynamical system, it is often the case that the only accessible information is a time series of some scalar function $\xi(\mathbf{x}(t))=\xi(t)$ of a d-dimensional state variable $\mathbf{x}(t)$. Using the delay coordinate embedding technique, Takens [9] showed that a delay coordinate vector

$$\mathbf{z}(t)=(\xi(t),\xi(t-T_D),\ldots,\xi(t-(n-1)T_D)),$$

with a conveniently chosen delay time T_D and a sufficiently large n, is generally a global one-to-one representation of the system state $\mathbf{x}(t)$. Using a Poincaré surface of section, one obtains a set of discrete state variables $\mathbf{z}_i=\mathbf{z}(t_i)$, where t_i denotes the time at the ith orbit crossing the surface of section. In the presence of parameter variation, delay coordinate embedding leads to a map that in general will depend on all parametric changes that were in effect in the time interval $t_i\leqslant t\leqslant t_i-nT_D$ [3,4], i.e., the state variable $\mathbf{z}_{i+1}$ must depend on not only the current value of the parameter p_i, but also r previous parameter values $p_{i-1},\ldots,p_{i-r}$. Noticing that $p_i,p_{i-1},\ldots,p_{i-r+1}$ remain fixed in the process from time i to time $i+1$, we get an $(n+r+1)$-dimensional map

$$\begin{aligned} \mathbf{z}_{i+1}&=\mathbf{f}(\mathbf{z}_i,p_{i-r},p_{i-r+1},\ldots,p_{i-1},p_i),\\ p_{i-r+1}&=p_{i-r+1},\\ &\vdots\\ p_{i-1}&=p_{i-1},\\ p_i&=p_i,\\ p_{i+1}&=g(\mathbf{z}_i,p_{i-r},p_{i-r+1},\ldots,p_{i-1},p_i), \end{aligned} \tag{2.1}$$

when the parameter p is activated. The Jacobian matrix of this map is

$$\mathbf{T}=\begin{pmatrix} \mathbf{D}_z\mathbf{f} & \mathbf{D}_{p_i-r}\mathbf{f} & \mathbf{D}_{p_{i-r+1}}\mathbf{f} & \cdots & \mathbf{D}_{p_{i-1}}\mathbf{f} & \mathbf{D}_{p_i}\mathbf{f} \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \mathbf{D}_z g & \dfrac{\partial g}{\partial p_{i-r}} & \dfrac{\partial g}{\partial p_{i-r+1}} & \cdots & \dfrac{\partial g}{\partial p_{i-1}} & \dfrac{\partial g}{\partial p_i} \end{pmatrix}, \tag{2.2}$$

where all the partial derivatives are evaluated at the fixed point $(\mathbf{z}_*,p_{0,}\ldots p_0)$. Note that the parameter variables in Eq. (2.1) are ordered with decreasing of subscripts so that $\mathbf{D}_z g,\partial g/(\partial p_i-r),\ldots,\partial g/\partial p_i$ appears in the last row of $\mathbf{T}$. This will benefit the description of the following equations.

One can see that when $r=0$ the map (2.1) and its Jacobian matrix return to the forms of known mathematical model described in Sec. I. Thus, without losing generality, we discuss only the case of using delay coordinates. Our aim is to determine suitable $\mathbf{D}_z g,\partial g/(\partial p_i-r),\ldots,\partial g/\partial p_i$ such that the eigenvalues of the $(n+r+1)$-dimensional matrix $\mathbf{T}$ have specified values. We limit the form of g to ensure that $(\mathbf{z}_*,p_{0,}\ldots p_{0,})$ is the fixed point of the

$(n+r+1)$-dimensional system. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+r+1}$ be $n+r+1$ eigenvalues of $\mathbf{T}$; then we have

$$|\mathbf{T}| = c_0,$$

$$\sum_{i=1}^{n+r+1} |\mathbf{T}_{\{i,i\}}| = c_1,$$

$$\sum_{i=1}^{n+r} \sum_{j=i+1}^{n+r+1} |\mathbf{T}_{\{i,i\},\{j,j\}}| = c_2, \tag{2.3}$$

$$\sum_{i=1}^{n+r-1} \sum_{j=i+1}^{n+r} \sum_{k=j+1}^{n+r+1} |\mathbf{T}_{\{i,i\},\{j,j\},\{k,k\}}| = c_3,$$

$$\vdots$$

$$\sum_{i=1}^{n+r+1} t_{ii} = c_{n+r},$$

where

$$c_{n+r} = \sum_{i=1}^{n+r+1} \lambda_i,$$

$$c_{n+r-1} = \sum_{i=1}^{n+r} \sum_{j=i+1}^{n+r+1} \lambda_i \lambda_j,$$

$$c_{n+r-2} = \sum_{i=1}^{n+r-1} \sum_{j=i+1}^{n+r} \sum_{k=j+1}^{n+r+1} \lambda_i \lambda_j \lambda_k,$$

$$\vdots$$

$$c_0 = \lambda_1 \lambda_2 \cdots \lambda_{n+r+1},$$

and $|\mathbf{T}_{\{i,j\},\{k,l\},\ldots}|$ represents the determinant obtained by eliminating the ith row and the jth column, the kth row and the lth column, etc. of the matrix $\mathbf{T}$ and t_{ij} denotes the element of $\mathbf{T}$.

In Ref. [5] we pointed out that the $n+r+1$ equations (2.3) include $n+r+1$ variables [i.e., $\mathbf{D_z}g, \partial g/(\partial p_{i-r}), \ldots, \partial g/\partial p_i$] to be determined and they are linear equations with respect to these variables. Based on this observation, we obtain the solution through algebra:

$$\left(\mathbf{D}_z g, \frac{\partial g}{\partial p_{i-r}}, \ldots, \frac{\partial g}{\partial p_i}\right)^T = \mathbf{A}^{-1}(c_0', c_1', c_2', \ldots, c_{n+r}')^T, \tag{2.4}$$

where

$$c_0' = c_0,$$

$$c_1' = c_1 - |\mathbf{T}_{\{n+r+1,n+r+1\}}|,$$

$$c_2' = c_2 - \sum_{i=1}^{n+r} |\mathbf{T}_{\{n+r+1,n+r+1\},\{i,i\}}|,$$

$$c_3' = c_3 - \sum_{i=1}^{n+r-1} \sum_{j=i+1}^{n+r} |\mathbf{T}_{\{n+r+1,n+r+1\},\{i,i\},\{j,j\}}|,$$

$$\vdots$$

$$c_{n+r}' = c_{n+r} - \sum_{i=1}^{n+r} t_{ii}$$

and

$$a_{1s} = (-1)^{n+r+s-1} |\mathbf{T}_{\{n+r+1,s\}}|,$$

$$a_{2s} = (-1)^{n+r+s-2} \sum_{i=1, i\neq s}^{n+r} |\mathbf{T}_{\{n+r+1,s\},\{i,i\}}|,$$

$$a_{3s} = (-1)^{n+r+s-3} \sum_{i=1; i\neq s}^{n+r-1} \sum_{j=i+1; j\neq s}^{n+r} |\mathbf{T}_{\{n+r+1,s\},\{i,i\},\{j,j\}}|,$$

$$a_{4s} = (-1)^{n+r+s-4} \sum_{i=1; i\neq s}^{n+r-2} \sum_{j=i+1; j\neq s}^{n+r-1} \sum_{k=j+1; k\neq s}^{n+r} \times |\mathbf{T}_{\{n+r+1,s\},\{i,i\},\{j,j\},\{k,k\}}|,$$

$$\vdots$$

$$a_{n+r+1,s} = \begin{cases} 0 & \text{for } s = 1,2,\ldots,n+r \\ 1 & \text{for } s = n+r+1, \end{cases}$$

in which a_{ij} is the element of the matrix $\mathbf{A}$.

Obviously, the pole placement problem has a unique solution if and only if the $(n+r+1)\times(n+r+1)$ matrix $\mathbf{A}$ is of rank $n+r+1$. This is also the controllability condition.

In the following we list some useful feedback forms for different purposes and derive the coefficients in them. In the case that the location of the desired orbit is given definitely and the prompt information is accessible, one can apply the feedback form (1.3), which is used in the original pole placement technique and the OGY method, to stabilize the unstable desired orbit. The feedback form (1.3) should be replaced by

$$p_{i+1} = p_0 + \mathbf{k}^T \cdot (\mathbf{z}_{i+1} - \mathbf{z}_*) + k_{n+1}(p_i - p_0) + \cdots + k_{n+r}(p_{i-r+1} - p_0) \tag{2.5}$$

in the case of using delay coordinates. However, this feedback forms always ensures that $|\mathbf{T}| \equiv 0$. To keep this relation valid, one of the $n+r+1$ eigenvalues, say, λ_{n+r+1}, must remain zero constantly. Thus one just needs to place the other $n+r$ eigenvalues in this case. The coefficients included in Eq. (2.5) can be easily achieved when $\mathbf{D}_z g, \partial g/(\partial p_{i-r}), \ldots, \partial g/\partial p_i$ are obtained. They are

$$\mathbf{k}=[(\mathbf{D}_z\mathbf{f})^T]^{-1}[\mathbf{D}_z g]^T,$$

$$k_{n+1}=\frac{\partial g}{\partial p_i}-\mathbf{k}^T\cdot\mathbf{D}_{p_i}\mathbf{f},$$

$$\vdots \tag{2.6}$$

$$k_{n+r}=\frac{\partial g}{\partial p_{i-r+1}}-\mathbf{k}^T\cdot\mathbf{D}_{p_{i-r+1}}\mathbf{f}.$$

By using the original pole placement technique one can approach the same aim (for the case of using the delay coordinate technique the readers may refer to Ref. [3]) in a different way. The difference is that the original method is used in an $(n+r)$-dimensional space.

When the prompt information is not accessible, one can use the previous coordinate $\mathbf{z}_i$ instead of $\mathbf{z}_{i+1}$ to construct the feedback form

$$p_{i+1}=p_0+\mathbf{k}^T\cdot(\mathbf{z}_i-\mathbf{z}_*)+k_{n+1}(p_i-p_0)+\cdots k_{n+r+1}(p_{i-r}-p_0). \tag{2.7}$$

This feedback form can also be used to stabilize the desired orbit $\mathbf{z}_*$ [5]. The solutions of the coefficients are rather simple:

$$\mathbf{k}=[\mathbf{D}_z g]^T,$$

$$k_{n+1}=\frac{\partial g}{\partial p_i},$$

$$\vdots \tag{2.8}$$

$$k_{n+r+1}=\frac{\partial g}{\partial p_{i-r}}.$$

For the purpose of finding the exact location of the desired orbit or going about tracking, one can apply

$$p_{i+1}=p_0+\mathbf{k}^T\cdot(\mathbf{z}_{i+1}-\mathbf{z}_i)+k_{n+1}(p_i-p_0)+\cdots k_{n+r+1}(p_{i-r}-p_0) \tag{2.9}$$

as the feedback form [5]. In this case we get

$$\mathbf{k}=[(\mathbf{D}_z\mathbf{f}-\mathbf{I}_n)^T]^{-1}[\mathbf{D}_z g]^T,$$

$$k_{n+1}=\frac{\partial g}{\partial p_i}-\mathbf{k}^T\cdot\mathbf{D}_{p_i}\mathbf{f},$$

$$\vdots \tag{2.10}$$

$$k_{n+r+1}=\frac{\partial g}{\partial p_{i-r}}-\mathbf{k}^T\cdot\mathbf{D}_{p_{i-r}}\mathbf{f},$$

where $\mathbf{I}_n$ is the $n\times n$ unit matrix. All the vectors $(\mathbf{k}^T, k_{n+1},\ldots,k_{n+r+1})$ suitable to stabilize $\mathbf{z}_*$ form a region in R^{n+r+1} space. We name the region the stability region of the desired orbit. For the purpose of application, we pay special attention to the so-called center point of the stability region, which corresponds to $\lambda_i=0$, $i=1,2,\ldots,n+r+1$. Since Eq. (2.9) does not include $\mathbf{z}_*$ explicitly, the perturbations with any point in the stability region can force the system trajectories to evolve towards the exact position of $\mathbf{z}_*$ even if it has not been determined precisely in the beginning. Based on the same reason, one can apply this kind of feedback function to go about tracking. That is, when the position of the desired orbit moves a small distance due to the change of the system parameter(s), the perturbations will drive the system trajectories to track the changed orbit. This tracking method is different from the previous tracking technique [6] based on the OGY method since it does not need to ''guess'' or ''predict'' the changed position of the desired objective in advance (interested readers are referred to Ref. [5] to find examples of going about tracking using this method).

III. APPLICATION

Stabilizing unstable orbit using delay information, finding the exact desired objective, and going about tracking are consequences of applying the extended technique. In this section we show how to target an unstable periodic orbit from the stable one created at the same tangent bifurcation point. At the time before targeting, the position of the unstable orbit is unknown. Thus those feedback forms that are constructed by use of the desired orbit explicitly, such as Eqs. (2.5) and (2.7), cannot be used for this purpose. One needs to realize the aim by applying the feedback form (2.9). Our approach is as follows. We first manage the system to lie on the stable state near the tangential bifurcation point. Then we can measure **A** and **B** at the stable state and calculate the coefficients in Eq. (2.9) according to Eqs. (2.4) and (2.10). Considering the symmetry of the stable and the unstable states, we derive a transformation law that transforms the coefficients obtained at the stable state to those suitable for stabilizing the unstable state. When using the transformed coefficients to the system, the perturbations can drive the system dynamical trajectories to depart from the stable state and terminate on the unstable one automatically.

The main advantage of this targeting method is that as the targeting procedure proceeds the stable state becomes unstable while the unstable one (i.e., the targeting objective) becomes stable and therefore one need not know the position of the unstable state in advance, which certainly benefits the experimentalists. This makes it different from the previous targeting method [7]. The following subsections realize the above idea in the cases of one- and high-dimensional systems, respectively.

A. One-dimensional systems

Let us consider the so-called sine-square map

$$z_{i+1}=A\sin^2(z_i-B), \tag{3.1}$$

which describes a hybrid optical bistability device using a twisted nematic liquid crystal as the nonlinear medium in the limit of a very long delay [11]. Figure 1 shows the bifurcation diagram in the z_i-A plane for fixed $B=3.0$, which exhibits a typical bistability phenomenon. In this figure the dashed line denotes the unstable steady states while the solid lines represent the stable steady states of the map. The steady states are the solutions of $z=A\sin^2(z-B)$. The two turning points of the stable and unstable states are tangential bifur-

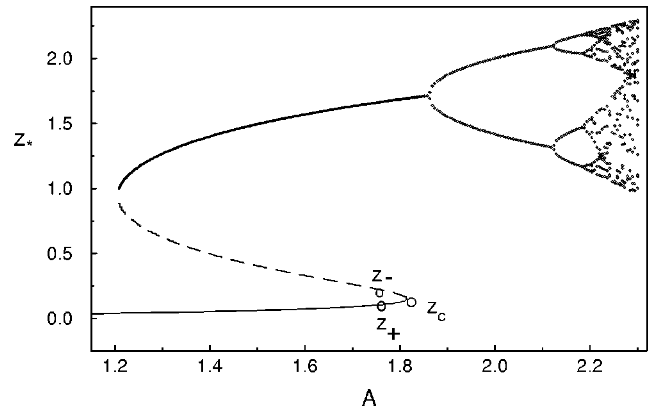

FIG. 1. The z_*-A bifurcation diagram for $B=3.0$ of the sine-square map, which shows typical bistability. The dashed line indicates the unstable state and z_c denotes a turning point.

cation points. In Fig. 1 z_c denotes one of the turning points. The parameter value of A at z_c is 1.815. Suppose that one intends to get the output corresponding to the unstable steady state z_- at $A=1.76$ from an experimental device and the present output corresponds to the stable steady state z_+ at the same parameter. Then what one needs to do is to target z_- from z_+ using the information obtained at z_+. We emphasize that the exact position of z_- is unknown in advance.

In order to demonstrate the principle of the method we consider only the case of $r=0$. In this case, for a one-dimensional system Eq. (2.10) appears as

$$k_1=-\left[c_0-c_1\frac{\partial f}{\partial z}+\left(\frac{\partial f}{\partial z}\right)^2\right]\Big/\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial z}-1\right),$$

$$k_2=\left(c_0-c_1+\frac{\partial f}{\partial z}\right)\Big/\left(\frac{\partial f}{\partial z}-1\right). \tag{3.2}$$

Let z_* be a fixed point. The boundaries of its stability region are determined by $|\lambda_1|=1$ or $|\lambda_2|=1$. It is not difficult to find that the stability region is a triangle, whose apexes are

$$(k_1,k_2)=\left[-\left(\frac{\partial f}{\partial z}-1\right)\Big/\frac{\partial f}{\partial p},1\right] \text{ for } (\lambda_1,\lambda_2)=(1,1),$$

$$(k_1,k_2)=\left[-\left(\frac{\partial f}{\partial z}+1\right)^2\Big/\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial z}-1\right),\right.$$

$$\left.\left(3+\frac{\partial f}{\partial z}\right)\Big/\left(\frac{\partial f}{\partial z}-1\right)\right]$$

$$\text{for } (\lambda_1,\lambda_2)=(-1,-1),$$

and

$$(k_1,k_2)=\left[-\left(\frac{\partial f}{\partial z}+1\right)\Big/\frac{\partial f}{\partial p},1\right]$$

$$\text{for } (\lambda_1,\lambda_2)=(1,-1) \text{ or } (\lambda_1,\lambda_2)=(-1,1).$$

The "center" point of the triangle, which is defined by $(\lambda_1,\lambda_2)=(0,0)$, is

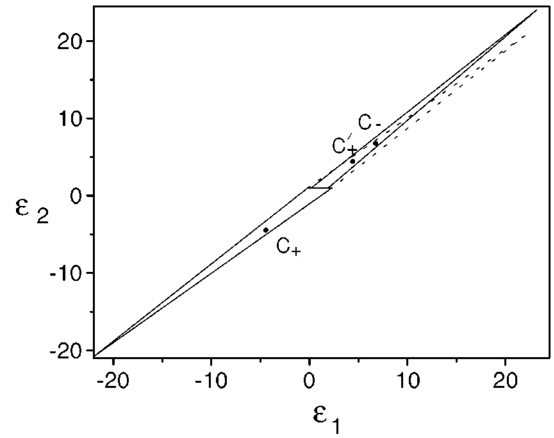

FIG. 2. Stable region of z_+ (the solid-line down triangle) and stable region of z_- (the solid-line up triangle. The dashed-line up triangle is transformed from the down triangle.

$$(k_1,k_2)=\left[-\left(\frac{\partial f}{\partial z}\right)^2\Big/\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial z}-1\right),\frac{\partial f}{\partial z}\Big/\left(\frac{\partial f}{\partial z}-1\right)\right].$$

In the neighborhood of z_c, assuming that the two points z_+ and z_- are located symmetrically with respect to z_c, we have $\partial f/\partial z=1-\delta$ at z_+ and $\partial f/\partial z=1+\delta$ at z_-, where δ is a small positive number. Then we can obtain the apexes and the center point of the stability region of z_+:

$$(k_1,k_2)=\left(\delta\Big/\frac{\partial f}{\partial p},1\right),$$

$$(k_1,k_2)=\left[-4\Big/\frac{\partial f}{\partial p}+4/\delta\frac{\partial f}{\partial p},-4/\delta+1\right], \tag{3.3}$$

$$(k_1,k_2)=\left(-2\Big/\frac{\partial f}{\partial p},1\right),$$

and

$$(k_1,k_2)=\left[-2\Big/\frac{\partial f}{\partial p}-1/-\delta\frac{\partial f}{\partial p},-1/\delta+1\right], \tag{3.4}$$

respectively. Obviously when replacing δ in the above equations with $-\delta$ we get the corresponding apexes and the center point of the stability region of z_-. Thus, when $\delta\to 0$, i.e., when z_+ and z_- approach z_c closely, we get a transformation law

$$(k_1,k_2)\to(-k_1,k_2),$$

$$(k_1,k_2)\to(-k_1,-k_2), \tag{3.5}$$

$$(k_1,k_2)\to(k_1,k_2)$$

for the apexes and

$$(k_1,k_2)\to(-k_1,-k_2) \tag{3.6}$$

for the center points of the two triangles.

Let us apply the result to the sine-square map (3.1). We take B as our control parameter, which can be adjusted around the nominal value 3.0 within $|\delta B|<0.1$. Figure 2

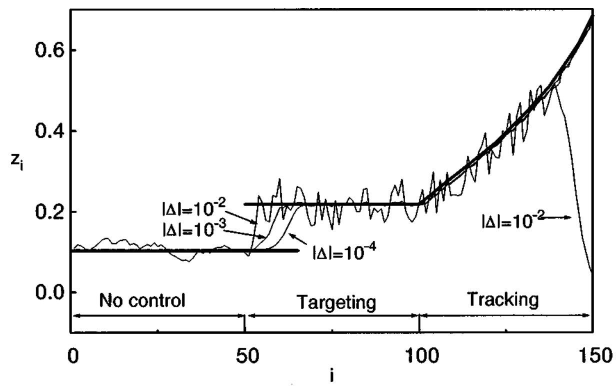

FIG. 3. Control results for the sine-square map. In the control section dynamical trajectories wander around z_+ with the evolution of time. The value of A in this section is 1.76. In the targeting section feedback following Eq. (2.9) is added to the system parameter B. The coefficients in Eq. (2.9) are obtained at z_+ according to the transformation law (3.6). The results indicate that the trajectories depart from z_+ and enter the neighborhood of z_- after certain times of iterates and larger noise interference may step up the targeting process. In this section the parameter A is kept as that in the no control section. In the tracking section the parameter A begins to change as $A_{i+1}=A_i-0.01$, while we keep the perturbations on B according to the feedback law (2.9) with the same coefficients as those in the targeting section. The results in this section show that by only applying the coefficients calculated at z_+ one can track the changing objective when the system parameter A changes in a wider parameter region even in the present of larger noise interference. The heavy lines indicate the exact positions of z_+ (in the no control section) and z_- (in the targeting section) and the changing objective (in the tracking section) and the values of $|\Delta|$ give the amplitudes of noise interfering with the system signals.

shows the stability region of z_+ (solid-line down triangle) and the stability region of z_- (solid-line up triangle). The dashed-line up triangle and its center point C'_+ are obtained by transforming the down triangle and its center point C_+, respectively, according to the transformation law. One can find that most of the two up triangles overlap, which verifies that the stability region of z_- can be obtained from that of z_+ approximately when z_+ and z_- approach the tangent bifurcation point. Since C'_+ is closest to the center point C_- and lies outside the stability region of z_+, one can expect that the state trajectory must depart from z_+ and evolve towards z_- automatically when one perturbs the system according to the feedback law (2.9) with the coefficients determined by C'_+.

Figure 3 shows the numerical results of applying Eq. (2.9) with coefficients determined by C'_+ to the system (3.1) under the assumption that the system is influenced by noise, i.e., we add white noise Δ_i to the system: $z_{i+1}=A\sin^2(z_i-B)+\Delta_i$. In the figure we provide several results obtained in different noise levels, where the values of $|\Delta|$ indicate noise amplitudes. One can see from the figure that the targeting procedure works very well even in the presence of small noise. After the trajectory settles down on z_-, the same feedback law with the same coefficients used in the targeting section can be applied to track the unstable state when the system parameter(s) change slowly as a function of time, as we have pointed out in Sec. II. Figure 3 also demonstrates the tracking results when A changes as $A_{i+1}=A_i-0.01$ in different noise levels. It is shown that the tracking method is effective in relatively large noise interference. Thus, by using the targeting method, we can switch the dynamics of the system between the stable state and the unstable state. Furthermore, by applying the tracking procedure we can obtain the outputs corresponding to the unstable states in a wider parameter interval.

B. High-dimensional systems

We shall discuss only the transformation law for the center points in this section. Let $\bar{\lambda}_i$, $i=1,2,\ldots,n$, denote the eigenvalues of $\mathbf{D}_z\mathbf{f}$. From the knowledge of matrix we have

$$\mathbf{D}_z\mathbf{f}=\mathbf{M}\begin{pmatrix}\bar{\lambda}_1 & 0 & \cdots & 0\\ 0 & \bar{\lambda}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\lambda}_n\end{pmatrix}\mathbf{M}^{-1}, \tag{3.7}$$

where $\mathbf{M}$ is the adjoint matrix of $\mathbf{D}_z\mathbf{f}$. It is easy to show that

$$\mathbf{D}_z\mathbf{f}-\mathbf{I}_n=\mathbf{M}\begin{pmatrix}\bar{\lambda}_1-1 & 0 & \cdots & 0\\ 0 & \bar{\lambda}_2-1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\lambda}_n-1\end{pmatrix}\mathbf{M}^{-1}, \tag{3.8}$$

where $\mathbf{I}_n$ is the $n\times n$ unit matrix. Let $\mathbf{M}^{-1}[\mathbf{D}_z g]^T \equiv (b_1,b_2,\ldots,b_n)^T$; we rewrite Eq. (2.9) as

$$\mathbf{k}^T=\begin{pmatrix}\frac{b_1}{\bar{\lambda}_1-1}\mathbf{M}_{11}+\frac{b_2}{\bar{\lambda}_2-1}\mathbf{M}_{12}+\cdots+\frac{b_n}{\bar{\lambda}_n-1}\mathbf{M}_{1n}\\ \frac{b_1}{\bar{\lambda}_1-1}\mathbf{M}_{21}+\frac{b_2}{\bar{\lambda}_2-1}\mathbf{M}_{22}+\cdots+\frac{b_n}{\bar{\lambda}_n-1}\mathbf{M}_{2n}\\ \vdots\\ \frac{b_1}{\bar{\lambda}_1-1}\mathbf{M}_{n1}+\frac{b_2}{\bar{\lambda}_2-1}\mathbf{M}_{n2}+\cdots+\frac{b_n}{\bar{\lambda}_n-1}\mathbf{M}_{nn}\end{pmatrix}, \tag{3.9}$$

$$k_{n+1}=\frac{\partial g}{\partial p_i}-\mathbf{k}^T\cdot\mathbf{D}_{p_i}\mathbf{f},$$

$$\vdots$$

$$k_{n+r+1}=\frac{\partial g}{\partial p_{i-r}}-\mathbf{k}^T\cdot\mathbf{D}_{p_{i-r}}\mathbf{f}.$$

Let $\mathbf{z}_+$ and $\mathbf{z}_-$ be a pair of fixed points located in symmetrical positions with respect to the tangential bifurcation point $\underline{\mathbf{z}_c}$ at which they are created. In the vicinity of $\mathbf{z}_c$ we have $\bar{\lambda}_1=1+\delta$ for the unstable point $\mathbf{z}_-$ and $\bar{\lambda}_1=1-\delta$ for the stable point $\mathbf{z}_+$, where δ is a small positive number. Thus Eq. (3.9) appears as

$$
\mathbf{k}^T=\begin{pmatrix}
\frac{b_1}{\pm\delta}\mathbf{M}_{11}+\frac{b_2}{\lambda_2-1}\mathbf{M}_{12}+\cdots+\frac{b_n}{\lambda_n-1}\mathbf{M}_{1n}\\
\frac{b_1}{\pm\delta}\mathbf{M}_{21}+\frac{b_2}{\lambda_2-1}\mathbf{M}_{22}+\cdots+\frac{b_n}{\lambda_n-1}\mathbf{M}_{2n}\\
\vdots\\
\vdots\\
\vdots\\
\frac{b_1}{\pm\delta}\mathbf{M}_{n1}+\frac{b_2}{\lambda_2-1}\mathbf{M}_{n2}+\cdots+\frac{b_n}{\lambda_n-1}\mathbf{M}_{nn}
\end{pmatrix}, \tag{3.10}
$$

$$k_{n+1}=\frac{\partial g}{\partial p_i}-\mathbf{k}^T\cdot\mathbf{D}_{p_i}\mathbf{f},$$

$$\vdots$$

$$k_{n+r+1}=\frac{\partial g}{\partial p_{i-r}}-\mathbf{k}^T\cdot\mathbf{D}_{p_{i-r}}\mathbf{f},$$

where plus and minus signs correspond to $\mathbf{z}_-$ and $\mathbf{z}_+$, respectively. Noticing that the items including the factor $1/\delta$ give the main contribution to the sums when $\delta\to 0$, we then obtain the transformation law for the two center points of the stability regions of $\mathbf{z}_+$ and $\mathbf{z}_-$:

$$\mathbf{k}\to-\mathbf{k},\mathbf{k}\in R^{n+r+1}. \tag{3.11}$$

To support this transformation law, let us consider the Hénon map [12]

$$x_{i+1}=a-x_i^2+by_i,$$

$$y_{i+1}=x_i \tag{3.12}$$

as an example. The map has two fixed points: $(x_\pm, y_\pm)=\{[b-1\pm\sqrt{(b-1)^2+4a}]/2, x_\pm\}$, which are created at $a_c=-(b-1)^2/4$ through a tangential bifurcation, and (x_+, y_+), which is stable while (x_-, y_-) is unstable. Choosing the parameter b as our control parameter, which can be adjusted around the nominal value $b_0=0.3$ within $|\delta b|<0.01$, and using the feedback

$$b_{i+1}=b_0+k_1(x_{i+1}-x_i)+k_2(y_{i+1}-y_i)+k_3(b_i-b_0), \tag{3.13}$$

we can calculate the center points of the stability regions of (x_+, y_+) and (x_-, y_-), respectively, according to Eq. (2.10). For example, at $a=-0.12$ they are $(k_1^+, k_2^+, k_3^+)=(-18, -6, -6)$ and $(k_1^-, k_2^-, k_3^-)=(22,6,8)$, respectively. It can be seen that $(-k_1^+, -k_2^+, -k_3^+)$ is very close to (k_1^-, k_2^-, k_3^-), which verifies that the center point of the stability region of (x_-, y_-) can be obtained approximately from that of (x_+, y_+) following the transformation law (3.11).

Figure 4 shows the numerical result of applying Eq. (3.13) with $(k_1, k_2, k_3)=(-k_1^+, -k_2^+, -k_3^+)$ to perturb the noise-influenced system (3.12). The noise is added to the system signals as $x_{i+1}=a-x_i^2+by_i+\Delta_{xi}$ and $y_{i+1}=x_i+\Delta_{yi}$, where Δ_{xi} and $\triangle_{yi}$ are the white noise and limited in the interval $(-|\Delta|, |\Delta|)$. The dynamical system is set on

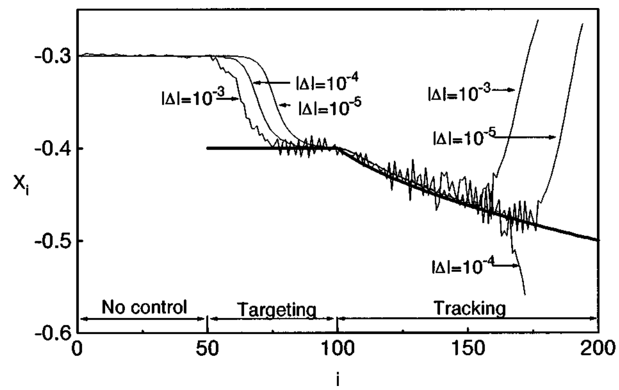

FIG. 4. Control results for the Hénon map: targeting x_- from x_+ and then tracking it when the parameter A changes as $A_{i+1}=A_i+0.0002$ in the case that the system signals are interfered with by additive noise. The procedure and the symbols in this figure are similar to those in Fig. 3.

(x_+, y_+) at $a=-0.12$ and $b=0.3$ in the beginning. The result indicates that the dynamical state trajectories are driven toward (x_-, y_-) by perturbations. Also, after the state trajectory settles down on (x_-, y_-), Eq. (3.13), with the same coefficients, can be used to track (x_-, y_-) when a changes.

IV. CONCLUSION

We finished the extension of the pole placement technique in the case of one adjustable system parameter. The extended method allows a more general choice of feedback functions. When using prompt linear feedback function, both our method and the original pole placement technique give the correct result, i.e., the original technique can be taken as a special case of ours.

The feedback function that does not include the desired orbit explicitly can be applied to find the exact location of the desired orbit and go about tracking when system parameters change slowly as well. In this paper we present an important application, i.e., targeting the unstable periodic orbit using the information of the corresponding stable periodic orbit created at the same tangential bifurcation point. The location of the unstable periodic orbit does not need to be known in advance. The feedback perturbations will drive the state trajectories to leave the stable orbit and terminate on the unstable one in the vicinity of the tangential bifurcation point. We emphasize that this method may be applied by experimentalists to realize the unstable outputs related to bistability from the experimental device even if the global mathematical model of the system is unknown.

Finally, we would like to emphasize the main difference among the original pole placement technique, the OGY method as well as its extensions, and the method in this paper. First of all, the first two methods only serve the prompt linear feedback form (1.3) or (2.5), which takes the state $\mathbf{z}_{i+1}$ and the desired orbit $\mathbf{z}_*$ as the feedback information. The method demonstrated in this paper is an extension of the pole placement technique that can be used to determine various feedback forms suitable for stabilizing unstable orbits and therefore it has wider applications. For example, it can be applied to calculate the coefficients in the feedback forms (2.7) and (2.9. As we have illustrated in Ref. [5] and also in the present paper, Eq. (2.7) can be used to stabilize the unstable periodic orbit by using the delayed information $\mathbf{z}_i$. While the feedback law (2.9), which is constructed by use of the prompt state $\mathbf{z}_{i+1}$ and the delayed state $\mathbf{z}_i$, provides us with a natural way of tracking the desired orbit when system parameters change as a time of function, a way to find the exact position of the desired orbit, and a technique to target the unstable periodic orbit from the stable one created at the same tangential bifurcation point in the case that the exact position of the unstable orbit is unknown in advance. In addition, the equations in this paper are derived using the delay coordinate embedding technique; therefore, they can be directly applied to experimental time series without any *a priori* knowledge of the system equations.

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